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## TWO NEW WORKS ON GRASSMANN'S GEOMETRICAL CALCULUS.\*

Until very recently, Schlegel's *Raumlehre*† was the only available introduction to the study of Grassmann's works. Two new works of a similar character, though both quite independent of Schlegel and of each other, are therefore a welcome addition to the rapidly increasing Grassmann literature. For it is well known that few works in the whole realm of mathematical literature present as serious difficulties to even the more advanced student as do Grassmann's writings.

Prof. Hyde's *Directional Calculus* may be recommended to all students of the higher mathematics as admirably suited to serve as a first introduction to the study of Grassmann's methods as applied to geometry. It is arranged on a plan similar to that of the elementary text-books on Quaternions, such as Hardy's, or Kelland and Tait's. This method of treatment and the insertion of a large number of exercises make the book well adapted for use in the class-room. In fact, such use seems to have been the principal object of the author. For the purpose of private study, the complete lack of references to the literature of the subject, even to Grassmann's own writings, must be regarded as a serious defect which detracts somewhat from its usefulness.

In connection with this, another objection might be raised against the general method of treatment followed by Prof. Hyde. The study of Grassmann should not be undertaken before the student has attained at least some familiarity with modern higher mathematics. But then, it becomes of prime importance to bring out fully the relation of the *Ausdehnungslehre* to the methods of modern Geometry and Algebra. The occasional hints given by Prof. Hyde in this direction are far from sufficient.

And here exception might also be taken to the title adopted by Prof. Hyde. "Directional Calculus" would seem to be a name very remote from Grassmann's own conception of his geometrical methods. It is only suggestive of the road that led Prof. Hyde himself to the study of Grassmann. "The Author, though formerly an enthusiastic admirer of Hamilton's Quaternions, has been brought, by study and experience in teaching both, to a firm belief in the great practical, as well as theoretical, superiority of Grassmann's system" (*Directional Calculus*, Preface, p. IV). But the *Ausdehnungslehre*,

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\* PEANO, GIUSEPPE, *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazioni della logica deduttiva*. Torino, Frat. Bocca, 1888. 8vo, pp. XII and 170.

HYDE, E. W., *The Directional Calculus*, based upon the methods of Hermann Grassmann. Boston, Ginn & Co., 1890. 8vo, pp. XII and 256.

† SCHLEGEL, VICTOR, *System der Raumlehre*. Leipzig, Teubner. 2 vols, 1872 and 1875.

even if restricted to its geometrical applications, is not merely a substitute for Quaternions. Hamilton's great creation was *a* method for studying geometrical questions; Grassmann's system gives us *the* geometrical calculus *par excellence*, of which all other methods are only special cases. Prof. Hyde himself seems to recognize this when he says (*l. c.*): "This superiority [of Grassmann's system] consists, according to the judgment of the writer, first, and largely, in the fact that Grassmann's system is founded upon, and absolutely consistent with, the idea of geometric dimensions. Second, in the fact that *all* geometric quantities appear as independent units, viz.: the point; the point at  $\infty$ , or line direction; the definite line; the line at  $\infty$ , or plane direction; the definite plane; and, finally, the plane at  $\infty$ , equivalent to a volume, which is a scalar. The same holds for space of any number of dimensions; in fact, it seems scarcely possible that any method can ever be devised, comparable with this, for investigating  $n$ -dimensional space."

Prof. Hyde's remarks on the comparative value of the methods of Hamilton and Grassmann are very just, and particularly valuable as coming from one who has made a thorough study of both.\* And yet, he does not seem to have freed himself entirely of the idea of *co-ordinating* the two systems.

The view taken by Prof. Peano of his subject is much broader. The Italian author defines it as follows (Preface, p. IV): "The geometrical calculus consists of a system of operations to be performed on geometric objects, analogous to those applied to numbers in Algebra. It gives us the means of expressing the results of geometrical constructions by means of formulæ, of representing geometrical propositions by means of equations, and of substituting transformations of equations to ordinary reasoning. The geometrical calculus has some analogy to analytic geometry; but it differs from it in the fact that, while in analytic geometry all calculations are performed on certain numbers that determine the geometrical objects, in this new science the calculations are performed on the objects themselves."

Prof. Peano is perhaps rather too sanguine in his expectations as to the future of the "new science." But his words are worth quoting (*ib.*, pp. VII, VIII): "In my opinion the time is not far when this geometrical calculus, or something analogous to it, will take the place of the methods actually in use in higher mathematical instruction. It is true, of course, that the study of this calculus, as that of any other science, requires a certain amount of time. But I do not think that more time is required for it than for the study of, say, the elements of analytic geometry. And it must be taken into account

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\* See his art. "Calculus of direction and position," in the *Am. Journ. Math.*, vol. VI, p. 1.

that the student will then be in possession of a method which comprises that of analytic geometry as a special case, is far more powerful, and lends itself admirably to the study of the geometrical applications of infinitesimal analysis, of mechanics and graphical statics; in fact, some parts of these sciences would be already known to one that had studied this calculus."

Very likely, Prof. Hyde would be the first to subscribe to this view; and he is certainly to be congratulated upon his persistent efforts in introducing the "new science" to English-speaking mathematicians by repeated courses of lectures on the subject in the University of Cincinnati, and finally by a very useful and (in the best sense of the term) *practical* text-book.

The first chapter of the "Directional Calculus" gives a succinct account of geometrical addition. It must be remembered that the idea of geometrical addition of points, vectors, areas, etc., which in its various forms was suggested by the geometrical illustration of complex numbers and by the requirements of mechanics, was developed independently of Grassmann, and partly before the publication of the first edition of his *Ausdehnungslehre* (1844), by Möbius, Bellavitis, Hamilton, Resal, and others. It is therefore only in the second chapter, on "Multiplication," that we are introduced to the conceptions and methods most characteristic of Grassmann's work. This rather long chapter (pp. 23 to 77) forms the basis of all that follows; it is the most important in the book, but at the same time the most difficult for the author to write, and for the beginner to read. On the whole, the task is well accomplished by Prof. Hyde. The method he follows in introducing the new ideas is often one of suggestion more than of actual rigid proof, and this has the advantage of leading the student more rapidly to an actual working knowledge of the subject. Some teachers will probably prefer to select only parts of this chapter, dealing as it does with so many unfamiliar ideas, before taking up Chapter III, "Applications to Plane Geometry." In Chapter IV, "Scalar Point Equations," the conic section is very fully treated from a point of view different from that adopted in the preceding chapter, the treatment in Chap. III corresponding to that by means of ordinary Cartesian co-ordinates, while the "scalar point equations" give the equivalent to the use of trilinear co-ordinates.

It is to be regretted that here, as well as in the following two chapters on Solid Geometry, the author confines himself entirely to equations of the second degree. Grassmann's own investigations on higher plane curves,\* and

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\* See his papers in Crelle's Journ., vols. 31 (1846), pp. 111-132; 36 (1848), pp. 177-182; 42 (1851), pp. 187-192; 52 (1856), pp. 254-275; and in *Gött. Nachr.*, 1872, p. 505.

his son's papers on twisted curves and the general theory of surfaces,\* would have furnished interesting material for an additional chapter.

The last chapter, "Applications to Statics," though very well written, is exceedingly brief. It is worth mentioning, however, that the elements of R. S. Ball's theory of screws have found a place in Prof. Hyde's book; and the readiness and naturalness with which Grassmann's system lends itself to the treatment of this beautiful theory is a further proof, if any be needed, of its wide usefulness.

In selecting English terms for Grassmann's new conceptions the author seems, in general, to have been judicious, making use of some terms that have become familiar to English readers through the quaternion calculus, such as vector, scalar, tensor, etc., and coining new expressions wherever it could not be avoided. Unfortunately, Prof. Hyde's terminology is widely different from that of Prof. Peano, of which more will be said below.

In outward appearance Prof. Hyde's book is very pleasing, both paper and printing being a credit to the publishing house (Ginn & Co.). The proof is read carefully, and but few misprints have been noticed. But the insertion of eight or ten blank pages at the end of every chapter gives the book a rather odd appearance. And why deprive the tetrahedra of their  $h$ 's? Why use the cumbersome symbol  $\sin < \begin{smallmatrix} \varepsilon_2 \\ \varepsilon_1 \end{smallmatrix}$  where  $\sin \varepsilon_1 \varepsilon_2$  would do just as well? Why say "bisectrix" for "bisector"? Such oddities do not add to the attractiveness of a book.

Prof. Peano's *Calcolo geometrico*, while pursuing the same object as the "Directional Calculus," is a work of a very different character. It is impossible in a brief review to give an adequate account of its varied and highly original contents.

The introduction, on the fundamental ideas and processes of Symbolic Logic, though it may deter some mathematicians from studying the main part of the work, is very interesting, in particular if read in connection with two little pamphlets by the author,† in which Boole's logical calculus is further developed and applied to the investigation of the principles of Algebra and Geometry. It is true that the philosophical form in which Grassmann himself presented his mathematical theories contributed in no small degree to

\* HERMANN GRASSMANN, [Jr.,] *Raumkurven*, 1886, and *Krumme Flächen*, 1888, Halle a. S. (School Programs).

† PEANO, JOSEPH, *Arithmetices principia nova methodo exposita*. Aug. Taur., Frat. Bocca, 1889. Pp. XVI and 20.

PEANO, GIUSEPPE, *I principii di geometria logicamente esposti*. Torino, Frat. Bocca, 1889. Pp. 40.

make his first work unpopular among mathematicians and, for a time, to prevent its exerting the stimulating influence on research the author had justly expected (see the preface to the second edition (1878) of the *Ausdehnungslehre* of 1844, p. XV). But it must not be forgotten that by severing Grassmann's geometrical theories from their logical foundation, much of their real value is necessarily lost.

Since the time of the first appearance of the *Ausdehnungslehre* (1844) great progress has been made in extending the bounds of mathematical speculation and gaining a broader philosophic view of the science of mathematics. The thorough investigation of the axioms of geometry has gradually led to the idea of  $n$ -dimensional space and non-Euclidean geometry, while an  $n$ -dimensional algebra has grown out of the study of the imaginary. At the same time, the true character of the symbolic language of mathematical formulæ has come to be better understood, owing to the development of a "calculus of operations" with so-called "symbolic" methods in various branches of mathematics, such as quaternions, determinants, invariants, differential equations, theory of groups, etc. Mathematicians have thus become familiar with a conception and use of symbols that hardly differs from that of Symbolic Logic.\* It may therefore be hoped that Prof. Peano's presentation of Boole's logical calculus may not be laid aside by the modern mathematician as something lying outside his sphere.

The plan adopted by Prof. Peano in presenting Grassmann's system of geometry may appear somewhat artificial at first glance. It has however great advantages for a rigorous and systematic development of the theory. The tetrahedral volume  $ABCD$  is taken as the fundamental geometric object. It is defined as a scalar quantity by taking into account the sense in which it is generated. The ratio of two such scalars being a mere number, two volumes can be defined as being equal if their ratios to the same third volume are equal; and the sum of two volumes  $A$  and  $B$  is defined as that volume  $I$  for which the numerical equation  $\frac{A}{Q} + \frac{B}{Q} = \frac{I}{Q}$  holds, where  $Q$  is any given unit-volume. Similarly, the more general equation  $\Sigma = mA + nB + pI + \dots$ , where  $m, n, p, \dots$  are numbers, can be interpreted as being equivalent to the numerical equation  $\frac{\Sigma}{Q} = m \frac{A}{Q} + n \frac{B}{Q} + p \frac{I}{Q} + \dots$ .

These definitions make it possible to apply to volumes certain operations

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\* See the interesting paper by Mr. A. B. Kempe, "Subject-Matter of Exact Thought," in *Nature*, vol. 43, No. 7 (Dec. 18, 1890), pp. 156-162.

that hold for numbers. For all other geometric objects, viz. *points*, "*lines*," and "*surfaces*," the rules of operation are derived by a very simple and elegant process from those originally established for volumes. The "*line*" is what Grassmann calls *Linientheil*, i. e. a localized vector (the "point-vector" of Prof. Hyde), while the "*surface*" is Grassmann's *Flächentheil*, or the "point-plane-vector" of Prof. Hyde.

Any expression of the form  $mA + nB + pC + \dots$  where  $m, n, p, \dots$  are numerical coefficients represents a *formazione geometrica*, let us say a *configuration*; it is said to be of the first, second, third, or fourth species, respectively, according as  $A, B, C, \dots$  are points, "*lines*," "*surfaces*," or volumes. The operations established for volumes are readily extended to configurations of all species; in addition, a new operation, Grassmann's "progressive" multiplication, is introduced. This forms the groundwork of the whole system.

The first three configurations are next taken up in separate chapters and fully discussed. It appears that a configuration of the first species in general represents a multiple point, but reduces to a *vector* (or point at infinity) whenever the sum of the coefficients of the configuration is  $= 0$ . In the configuration of the second species, the product of two vectors, or what amounts to the same, the sum of two equal and opposite "*lines*," plays a part analogous to that of the vector in the configuration of the first species. It is called a *bivector* by Prof. Peano ("plane-vector" by Prof. Hyde). It can be said that every configuration of the second species is in general reducible to a "*line*" and a "*bivector*," the latter being equivalent to a "*line*" at infinity. Similarly, in the configuration of the third species, the product of three vectors, or *trivector*, corresponds to a "*surface*" at infinity.

After the discussion of the configurations, a chapter each is devoted to the systems of the line, of the plane, and of space. Numerous "*applications*," or exercises, are appended to each chapter.

The last 45 pages of the book contain a chapter on the application of infinitesimal analysis to the geometrical configurations, and a chapter on linear transformations, with special reference to their application in geometry. This constitutes perhaps the most interesting, because the most original, part of Prof. Peano's work. A reference to the same author's larger treatise on the applications of the infinitesimal calculus to geometrical and mechanical investigations\* must here suffice.

ALEXANDER ZIWET.

JAN. 31, 1891.

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\* PEANO, GIUS., Applicazioni geometriche del calcolo infinitesimale. Torino, Frat. Bocca, 1887. Pp. XII and 336.